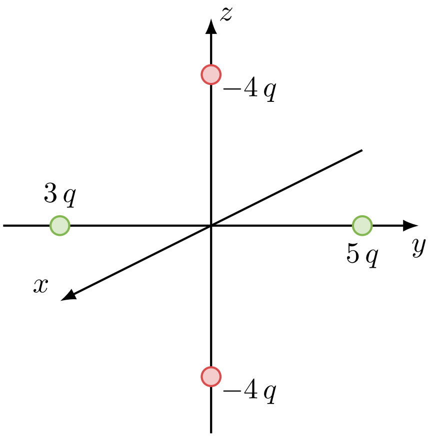
Homework 8: The Multipole Expansion
Due Monday, October 30

Problem 1: Approximate potential

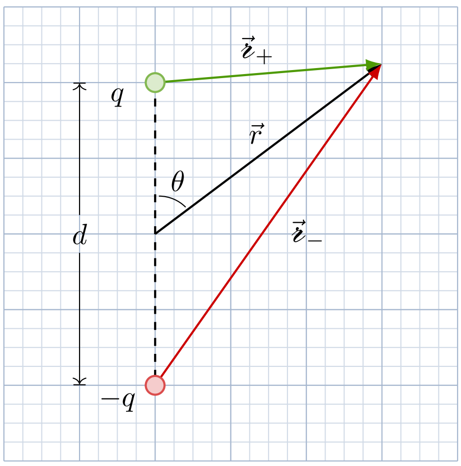
Four particles (one of charge $3q$, one of charge $5q$, and two of charge $-4q$) are placed as shown in the figure below:



Each charge is the same distance L from the origin. Find a simple approximate formula for the potential that is valid at points far from the charges ($r \gg L$). Express your answer in spherical coordinates. (By 'simple approximate formula', I mean the first non-zero term in the multipole expansion of the potential.)

Problem 2: Quadrupole and Octopole terms for a physical dipole

Two charges, q and $-q$, are separated by a distance d as in the figure below:



The potential at the point $\vec{r}$ is

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right) \,. \tag{1}$$

with

$$\mathscr{r}_{\pm} = \sqrt{r^2 + d^2 \mp 2\,rd\cos\theta}\ . \tag{2}$$

Taylor expand $1/\mathscr{r}_+$ and $1/\mathscr{r}_-$ out to order $(d/r)^3$ for $r \gg d$, write out $V(\vec{r})$ to that order, and identify the monopole (r^{-1}), dipole (r^{-2}), quadrupole (r^{-3}), and octopole (r^{-4}) terms in the potential. Do not do any integrals, just Taylor expand! (Keep in mind that the leading term in $1/\mathscr{r}_{\pm}$ for $r \gg d$ is just $1/r$, so the term of order $(d/r)^3$ is the *fourth* term in the Taylor expansion.)

Problem 3: Force on a point charge due to a pure dipole

A "pure" dipole p is situated at the origin, pointing in the z direction: $\vec{p} = p\,\hat{z}$. We worked out the potential and electric field for this dipole in class (and in the notes).

(a) What is the force on a point charge q located at $(0, 0, d)$ in Cartesian coordinates?

(b) What is the force on q if it is located at $(d, 0, 0)$?

(c) How much work does it take to move the charge q from $(0, 0, d)$ to $(d, 0, 0)$?

Problem 4: Multipole expansion for a line charge

A thin insulating rod, running from $z = -L/2$ to $z = +L/2$, carries a line charge $\lambda(z)$. In the following cases, find the leading (first non-zero) term in the multipole expansion of the potential.

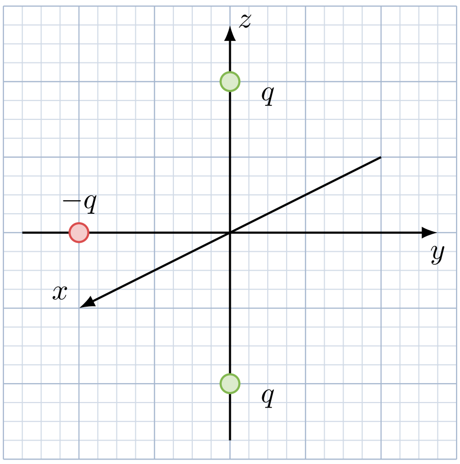
(a) $\lambda(z) = \lambda_0 \cos(\pi z/L)$

(b) $\lambda(z) = \lambda_0 \sin(2\pi z/L)$

(c) $\lambda(z) = \lambda_0 \cos(2\pi z/L)$

Problem 5: Multipole expansion for three point charges

The figure below shows three point charges, each a distance L from the origin:



Find the approximate *electric field* at points far from the charges. Express your answer in spherical coordinates, and include the contributions from the first two non-zero terms in the multipole expansion of the potential.